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# A CRACKED BEAM OR PLATE TRANSVERSELY LOADED BY A STAMP†

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**Abstract**—In this paper the problem of an infinite elastic beam or a plate containing a crack is considered. The medium is loaded transversely through a stamp which may be rigid or elastic. The problem is a coupled crack-contact problem which cannot be solved by treating the crack and contact problems separately and by using a superposition technique. First the Green's functions for the general case are obtained. Then the integral equations for a cracked infinite strip loaded by a frictionless stamp are obtained. With the question of fracture in mind, the primary interest in the paper has been in calculating the stress intensity factors. The results are given for a rigid flat stamp with sharp edges and for an elastic curved stamp. The effect of friction at the supports on the stress intensity factors is also studied and a numerical example is given.

## 1. INTRODUCTION

In this paper we consider the problem for a beam or a plate which contains a crack perpendicular to its boundaries and which is subjected to a symmetric transverse loading. The specific problem of interest is that of a beam supported at two points and loaded transversely through a rigid or an elastic stamp (see Fig. 1). The problem differs from the standard cracked strip problem considered, e.g. in [1–7] in that it is a coupled crack-contact problem in which the distribution of the transverse loads is not known and is dependent on the geometry of the crack as well as that of the stamp. Therefore, the routine superposition technique of calculating the crack surface tractions from the uncracked strip and using them to solve a perturbation problem in the cracked strip is not applicable.

In solving the beam problems another point of practical interest is the estimation of the effect of friction which may exist at the supports. This effect may be taken into consideration in formulating the problem by simply assuming that on the boundaries the tangential as well as the normal tractions are prescribed.

The solution of the problem is given for two stamp geometries, namely a rigid flat-ended stamp with sharp corners and a curved elastic stamp. Without the support friction and for small contact area under the stamp, the problem reduces to the three point bending problem for a beam which is considered in [5].

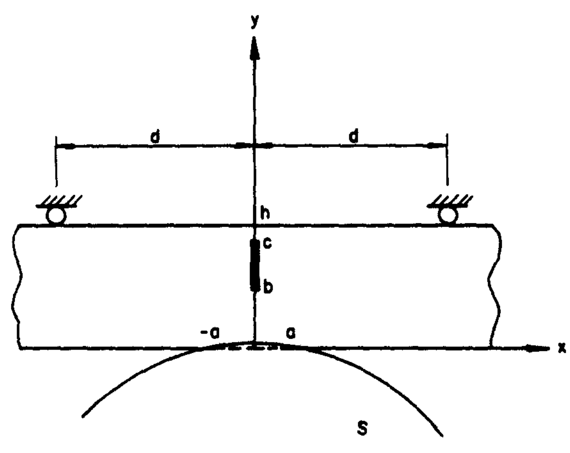


Fig. 1. An elastic strip containing a crack which is loaded through a stamp.

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## 2. INTEGRAL EQUATIONS OF THE PROBLEM

Consider the plane elasticity problem for an infinite strip shown in Fig. 2. Let $x = 0$ be a plane of symmetry. In addition to the tractions

$$
\begin{aligned}
\sigma_{yy}(x, 0) &= -p\delta(x - x_0) - p\delta(x + x_0),\\
\sigma_{xy}(x, 0) &= q\delta(x - x_0) - q\delta(x + x_0),\\
\sigma_{yy}(x, h) &= -P\delta(x - d) - P\delta(x + d),\\
\sigma_{xy}(x, h) &= Q\delta(x - d) - Q\delta(x + d),
\end{aligned}
\tag{1a–d}
$$

let the strip contain an "edge dislocation" given by

$$
\frac{\partial}{\partial y} u(o, y) = f\delta(y - y_0), \qquad 0 < y_0 < h, \tag{2}
$$

where $u$ is the $x$ component of the displacement vector. The solution of the problem under the "external loads" (1) and (2) would provide the necessary Green's functions to express the integral equations of a relatively general crack-contact problem in which there may be any number of cracks on the $x = 0$ plane and any number of stamps along the $y = 0$ and $y = h$ planes. Since the formulation of the problem is quite straightforward, in this paper most of the details regarding the derivation will be omitted. The displacement field in the strip may be expressed by the following Fourier integrals (see, e.g. [4, 8])

$$
\begin{aligned}
u(x, y) = \frac{2}{\pi}\int_0^\infty &\left\{\frac{1}{\alpha}\left(f_1 + \frac{\kappa+1}{2} g_1 + \alpha y g_2\right)\sinh \alpha y\right.\\
&\left.-\frac{1}{\alpha}\left(f_2 + \frac{\kappa+1}{2} g_2 + \alpha y g_1\right)\cosh \alpha y\right\}\sin \alpha x \, d\alpha\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\beta)\left(\frac{\kappa+1}{2} + |\beta| x\right) e^{-|\beta|x + i\beta y}\, d\beta,\\
v(x, y) = -\frac{2}{\pi}\int_0^\infty &\left\{\frac{1}{\alpha}\left(f_2 - \frac{\kappa-1}{2} g_2 + \alpha y g_1\right)\sinh \alpha y\right.\\
&\left.+\frac{1}{\alpha}\left(f_1 - \frac{\kappa-1}{2} g_1 + \alpha y g_2\right)\cosh \alpha y\right\}\cos \alpha x \, d\alpha\\
&+\frac{i}{2\pi}\int_{-\infty}^{\infty} A(\beta)\left(\frac{|\beta|}{\beta}\frac{\kappa-1}{2} - \beta x\right) e^{-|\beta|x + i\beta y}\, d\beta,
\end{aligned}
\tag{3a, b}
$$

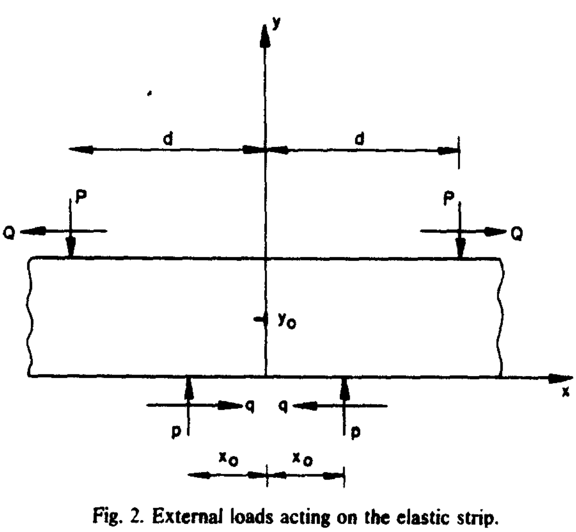


Fig. 2. External loads acting on the elastic strip.

where the functions $f_1(\alpha)$, $f_2(\alpha)$, $g_1(\alpha)$, $g_2(\alpha)$ and $A(\beta)$ are unknown. Equations (3) satisfy the symmetry condition given by

$$\tau_{xy}(0, y) = 0, \qquad 0 < y < h. \tag{4}$$

The five unknown functions which appear in (3) can be determined by using the five conditions given by (1) and (2). Using the stress-displacement relations, from (1)–(3) we obtain

$$A(\beta) = \frac{2}{i(1+\kappa)\beta} f\, e^{-i\beta y_0},$$

$$f_1(\alpha) = \frac{1}{D(\alpha)}\{-[\alpha h + \cosh\alpha h \sinh\alpha h]A_1 - \alpha^2 h^2 A_2 + [\sinh\alpha h + \alpha h\cosh\alpha h]A_3 - \alpha h \sinh\alpha h A_4\},$$

$$f_2(\alpha) = A_1,$$

$$g_1(\alpha) = \frac{1}{D(\alpha)}\{[\alpha h + \cosh\alpha h \sinh\alpha h]A_1 + \sinh^2\alpha h\, A_2 - [\sinh\alpha h + \alpha h\cosh\alpha h]A_3 + \alpha h \sinh\alpha h\, A_4\},$$

$$g_2(\alpha) = \frac{1}{D(\alpha)}\{-\sinh^2\alpha h\, A_1 + [\alpha h - \cosh\alpha h\sinh\alpha h]A_2 + \alpha h\sinh\alpha h\, A_3 + [\sinh\alpha h - \alpha h\cosh\alpha h]A_4\}, \tag{5a–e}$$

$$D(\alpha) = \sinh^2\alpha h - \alpha^2 h^2, \tag{6}$$

$$A_1 = \frac{1}{2\mu} p\cos\alpha x_0 + \frac{1}{1+\kappa} f y_0 \alpha\, e^{-\alpha y_0},$$

$$A_2 = \frac{1}{2\mu} q\sin\alpha x_0 - \frac{1}{\kappa+1} f(1-\alpha y_0)\, e^{-\alpha y_0},$$

$$A_3 = \frac{1}{2\mu} P\cos\alpha d - \frac{1}{1+\kappa} f\alpha(h-y_0)\, e^{-\alpha(h-y_0)},$$

$$A_4 = \frac{1}{2\mu} Q\sin\alpha d - \frac{1}{1+\kappa} f[1-\alpha(h-y_0)]\, e^{-\alpha(h-y_0)}, \tag{7a–d}$$

where $\mu$ is the shear modulus and $\kappa = 3-4\nu$ for plane strain (i.e. for plates) and $\kappa = (3-\nu)/(1+\nu)$ for plane stress (i.e. for beams), $\nu$ being the Poisson's ratio.

Now, let us assume that the strip contains cracks along a portion $L$ of the $x = 0$ plane and is loaded by stamps on the boundaries $y = 0$ and $y = h$, $M_0$ and $M_h$ corresponding to the respective contact areas. This means that $\sigma_{xx}$ is prescribed on $L$ and generally the displacements $u$ and $v$ are prescribed on $M_0$ and $M_h$. Then, substituting from (5)–(7) into (3) and using the appropriate stress-displacement relation we could obtain a system of five integral equations for the unknown functions $f(x)$, $q(x)$, $p(x)$, $P(x)$ and $Q(x)$. However, in order to simplify the problem, in this paper it will be assumed that at $y = h$ the tractions $P(x)$ and $Q(x)$ rather than the displacements $u$ and $v$ are prescribed and the tractions $p$ and $q$ are not independent. That is, either the stamps acting on $y = 0$ are frictionless, (i.e. $q = 0$) or the coefficient of friction on the contact area is constant. Therefore, the problem has only two unknown functions, $f(y)$ and $p(x)$, with $\sigma_{xx}$ and $v$ being prescribed on $L$ and $M_0$, respectively.

To complete the formulation of the problem, the contribution of the elastic stamp has to be incorporated into the integral equations. Considering only the curved elastic stamps which are in "smooth" contact with the strip and assuming that they have relatively large local radii of curvature, the local displacements in the stamps in the neighborhood of the contact area may be approximated by the standard half plane solution[9]. Referring to, for example[8], under the tractions $p(x)$ and $q(x)$, the derivative of the normal displacement in the stamp may be expressed as

$$\frac{\partial}{\partial x} v_s(x,-0) = -\frac{\kappa_s - 1}{4\mu_s} q(x) - \frac{1}{\pi}\frac{\kappa_s + 1}{4\mu_s}\int_{M_0} \frac{p(x_0)}{x_0 - x}\,dx_0, \tag{8}$$

where the subscript $s$ refers to the quantities in the stamp. The integral equation giving the contact pressure $p$ may then be obtained from

$$\frac{\partial}{\partial x}[v(x,+0) - v_s(x,-0)] = V(x) = \frac{d}{dx} v_0(x), \tag{9}$$

where $v_0(x)$ describes the profile of the stamp.

Expressing now $\sigma_{xx}(0,y)$ and $(\partial/\partial x)v(x,+0)$ in terms of the unknown functions $p$, $q$, and $f$ and using (8) and (9), after somewhat lengthy but straightforward analysis, we obtain

$$-\pi\beta_1 q(x) + \int_{M_0}\frac{p(t)}{t-x}\,dt + \beta_2\int_{M_0}[k_1(x,t)p(t) - k_2(x,t)q(t)]\,dt$$
$$+\beta_2\int_{M_h}[k_3(x,t)P(t) - k_4(x,t)Q(t)]\,dt + \beta_3\int_L f(t)\left[-\frac{4xt^2}{(x^2+t^2)^2}\right.$$
$$\left. + k_5(x,t)\right]dt = \pi\beta_3 V(x), \qquad x \in M_0, \tag{10}$$

$$\int_L f(t)\left[\frac{1}{t-y} + k_6^s(y,t) + k_6(y,t)\right]dt + \frac{\kappa+1}{4\mu}\int_{M_0}\{[k_7^s(y,t) + k_7(y,t)]p(t)$$
$$+[k_8^s(y,t) + k_8(y,t)]q(t)\}\,dt + \frac{\kappa+1}{4\mu}\int_{M_h}\{[k_9^s(y,t) + k_9(y,t)]P(t)$$
$$+[k_{10}^s(y,t) + k_{10}(y,t)]Q(t)\}\,dt = \frac{\kappa+1}{4\mu}\sigma_{xx}(0,y), \qquad y \in L, \tag{11}$$

where

$$\beta_1 = \frac{(\kappa-1)\mu_s - (\kappa_s-1)\mu}{(\kappa+1)\mu_s + (\kappa_s+1)\mu}, \qquad \beta_2 = \frac{(\kappa+1)\mu_s}{(\kappa+1)\mu_s + (\kappa_s+1)\mu},$$
$$\beta_3 = \frac{4\mu\mu_s}{(\kappa+1)\mu_s + (\kappa_s+1)\mu}, \tag{12}$$

the kernels $k_1, \ldots, k_{10}$ are bounded in their respective closed intervals and are given in Appendix A and $k_i^s(y,t)$, $(i = 6, \ldots, 10)$ represents the part of the kernel which becomes unbounded as $y$ and $t$ go to an end point $y = 0$ or $y = h$ simultaneously. Separation of these singular kernels is essential for an accurate treatment of the edge cracks. The kernels $k$ and $k^s$ are separated through the asymptotic analysis of the infinite integrals giving the sum of the two. For example, in $k_6^s + k_6$ the first term in these integrals reads as

$$H_1(y,t) = \int_0^\infty \frac{e^{2\alpha h} + 4\alpha h - e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2h^2 - 2 + e^{-2\alpha h}}\, t\alpha e^{-\alpha t}\sinh\alpha y\,d\alpha. \tag{13}$$

By adding and subtracting the asymptotic value of the integrand under the integral sign and evaluating the asymptotic integral, (13) may be expressed as follows:

$$H_1(y,t) = \frac{t}{2}\left[\frac{1}{(t-y)^2} - \frac{1}{(t+y)^2}\right] + \int_o^\infty \frac{4\alpha^2h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}}{e^{2\alpha h} - 4\alpha^2h^2 - 2 + e^{-2\alpha h}}\, t\alpha e^{-\alpha t}\sinh\alpha y\,d\alpha. \tag{14}$$

After evaluating all such terms and combining we obtain

$$k_6^s(y,t) = -\frac{1}{t+y} + \frac{6y}{(t+y)^2} - \frac{4y^2}{(t+y)^3} - \frac{1}{t-(2h-y)} - \frac{6(h-y)}{[t-(2h-y)]^2} - \frac{4(h-y)^2}{[t-(2h-y)]^3} - \frac{2}{2h-t+y} + \frac{4h+y-t}{(2h-t+y)^2} - \frac{12h(h-t)+4hy}{(2h-t+y)^3} + \frac{24hy(h-t)}{(2h-t+y)^4}, \tag{15}$$

$$k_7^s(y,t) = -\frac{y}{y^2+t^2} + \frac{y(y^2-t^2)}{(y^2+t^2)^2}, \tag{16}$$

$$k_8^s(y,t) = -\frac{2t}{y^2+t^2} + \frac{2ty^2}{(y^2+t^2)^2}, \tag{17}$$

$$k_9^s(y,t) = \frac{-(h-y)}{(h-y)^2+t^2} + \frac{(h-y)[(h-y)^2-t^2]}{[(h-y)^2+t^2]^2}, \tag{18}$$

$$k_{10}^s(y,t) = \frac{2t}{(h-y)^2+t^2} - \frac{2(h-y)^2 t}{[(h-y)^2+t^2]^2}. \tag{19}$$

In deriving the integral eqns (10) and (11) and the expressions for the kernels given in Appendix A, the following symmetry conditions have been used: The contact areas $M_0$ and $M_h$ are symmetric with respect to $x=0$ plane, $p(x)$, $P(x)$ and $v_0(x)$ are even functions of $x$, and $q(x)$ and $Q(x)$ are odd functions of $x$. The static equilibrium of the strip requires that

$$\int_{M_0} p(x)\,dx = \int_{M_h} P(x)\,dx. \tag{20}$$

Also, referring to the definition of the density function $f(t)$ given by (2), it is clear that, for example, for an imbedded crack along $(x=0,\ 0<b<y<c<h)$ $f$ must satisfy the following single-valuedness condition:

$$\int_b^c f(t)\,dt = 0. \tag{21}$$

The kernels $k_1, \ldots, k_{10}$ which appear in the integral eqns (10) and (11) are technically bounded in the respective closed domains of definition of their arguments and hence may be evaluated numerically without any difficulty. In this problem a Gauss–Legendre quadrature formula is used to evaluate the related infinite integrals. However, since the integrands have a singularity at $\alpha = 0$, considered individually most of these integrals are divergent. Expanding the integrands around $\alpha = 0$, the divergent part of the integrals can be separated. By using the equilibrium condition (20), it can then be shown that the sum of the divergent parts of the kernels is zero. Even though somewhat lengthy, this procedure is necessary for the accurate evaluation of the kernels.

## 3. SOLUTION FOR A RIGID FLAT STAMP

Let the beam or the plate have frictionless simple supports at $x = \mp d$, $y = h$ and be loaded through a frictionless rigid flat stamp of width $2a$ having sharp corners at $x = \mp a$. Assume that a through crack is located on $x = 0$, $b < y < c$. For this problem the integral eqns (10) and (11) are valid with

$$q(x)=0,\quad Q(x)=0,\quad P(x)=P\delta(x-d)+P\delta(x+d),\quad V(x)=0;$$
$$L=(b,c),\quad M_o=(-a,a),\quad \sigma_{xx}(0,y)=0,\quad b<y<c;$$
$$\beta_2 = 1,\quad \beta_3 = \frac{4\mu}{1+\kappa}, \tag{22}$$

where $P$ is the load at the supports $x = \mp d$ per unit thickness. After normalizing the intervals $(b, c)$ and $(-a, a)$ through the transformations

$$x = ar_1, \quad t = as_1, \quad (-a < t < a), \quad y = \frac{c-b}{2} r_2 + \frac{c+b}{2}, \quad t = \frac{c-b}{2} s_2 + \frac{c+b}{2}, \quad (b < t < c), \tag{23}$$

the singular integral eqns (10) and (11) may be solved numerically by using the Gauss–Chebyshev integration formulas [10]. Noting that the index of both equations is +1, the solution may be expressed as

$$\frac{p(t)}{2P/a} = f_1(s_1) = F_1(s_1)(1 - s_1^2)^{-1/2}, \tag{24}$$

$$\frac{\beta_3 f(t)}{2P/a} = f_2(s_2) = F_2(s_2)(1 - s_2^2)^{-1/2}. \tag{25}$$

Also, from (10), (11) and (22) observing that

$$\int_{M_k} k_3(x, t) P(t)\, dt = [k_3(x, d) + k_3(x, -d)] P = \frac{1}{2}\int_{-a}^{a} [k_3(x, d) + k_3(x, -d)] p(t)\, dt, \tag{26}$$

$$\int_{M_k} [k_9^s + k_9] P(t)\, dt = \frac{1}{2}\int_{-a}^{a} [k_9^s(x, d) + k_9^s(x, -d) + k_9(x, d) + k_9(x, -d)] p(t)\, dt, \tag{27}$$

the integral equations (10) and (11) may be expressed as follows:

$$\int_{-1}^{1} \sum_{j=1}^{2} k_{ij}(r, s) f_j(s)\, ds = 0, \qquad i = 1, 2, -1 < r < 1. \tag{28}$$

In (28) since all variables are defined in the same interval $(-1, 1)$, the subscripts in $r$ and $s$ have been deleted. The integral equations must be solved under the conditions

$$\int_{-1}^{1} f_1(s)\, ds = 1, \quad \int_{-1}^{1} f_2(s)\, ds = 0. \tag{29a, b}$$

The system of singular integral equations (28) are solved numerically by replacing (28) and (29) by [10–12]

$$\sum_{j=1}^{2} \sum_{k=1}^{n} k_{ij}(r_m, s_k) F_j(s_k) W_k = g_i(r_m), \quad i = 1, 2; \quad m = 1, \ldots, n-1, \tag{30}$$

$$\sum_{k=1}^{n} W_k F_1(s_k) = 1, \quad \sum_{k=1}^{n} W_k F_2(s_k) = 0 \tag{31a, b}$$

where

$$s_k = \cos\left(\frac{k-1}{n-1}\pi\right), \quad k = 1, \ldots, n, \quad r_m = \cos\left(\frac{2m-1}{2n-2}\pi\right), \quad m = 1, \ldots, n-1,$$

$$W_1 = W_n = \frac{\pi}{2(n-1)}, \quad W_k = \frac{\pi}{n-1}, \quad k = 2, \ldots, n-1. \tag{32}$$

Equations (30) and (31) give unknowns $F_j(s_k)$, $j = 1, 2$; $k = 1, \ldots, n$.

With the fracture of the beam or plate in mind, in the problems considered in this paper, the main interest is the evaluation of the stress intensity factors which are defined by

$$k(b) = \lim_{y\to b} [2(b-y)]^{1/2}\sigma_{xx}(0, y) = \lim_{y\to b} [2(y-b)]^{1/2} \frac{4\mu}{1+\kappa} f(y),$$

$$k(c) = \lim_{y\to c} [2(y-c)]^{1/2}\sigma_{xx}(0, y) = -\lim_{y\to c} [2(c-y)]^{1/2} \frac{4\mu}{1+\kappa} f(y). \qquad (33a, b)$$

Referring to (25), after solving the integral equations $k(b)$ and $k(c)$ may be evaluated from

$$k(b) = (2P/a)\sqrt{[(c-b)/2]}\, F_2(-1),$$

$$k(c) = -(2P/a)\sqrt{[(c-b)/2]}\, F_2(1). \qquad (34a, b)$$

When $c = h$, the crack becomes an edge crack. In this case the singular behavior of the solution was discussed in detail in[4]. Here it is sufficient to mention that the generalized Cauchy kernel found in this paper is identical to that of [4]. The numerical solution is carried out by letting $c = h$ and again using eqns (30)–(32). Needless to say, in this case the condition (29b) and hence the eqn (31b) is not valid. Also, at the end point $y = c = h$ the power of singularity of the density function $f(y)$ is zero (rather than $-(1/2)$), that is $f(h)$ is finite. Thus, $f(y)$ may still be defined by (25) and (30)–(32) may still be used to solve the problem provided (31b) is replaced by

$$F_2(1) = 0. \qquad (35)$$

The numerical results found for a cracked beam or plate loaded by a flat-ended rigid stamp are shown in Tables 1–5. The tables give the normalized stress intensity factors defined by

$$k^* = \frac{k}{(2P/h)\sqrt{(h-b)}} \text{ for the edge crack,}$$

$$k_0 = \frac{k}{(2P/h)\sqrt{[(c-b)/2]}} \text{ for the internal crack.}$$

Table 1 shows the edge crack results for the stamp width $a/h = 0.01$ which essentially corresponds to the three point loading problem. The table also shows the results given in[5] obtained for the three point loading problem by the method of boundary collocation. It may be seen that the agreement is quite good.

As the width of the rigid stamp $2a$ is increased, physically it would be expected that because of the "bending" of the strip the contact pressure at $x = 0$, $y = 0$ would decrease and eventually the surfaces would separate. This may be observed from Tables 2 and 3 and Figs. 3 and 4. Figures 3 and 4 show the distribution of the normalized pressure $p(x)/(2P/a)$. Both figures refer

Table 1. Stress intensity factor in a strip with an edge crack and loaded by a rigid flat stamp. $a/h = 0.01$, $k^*(b)$: present results, $\bar{k}^*(b)$: Ref.[5]

| | $d/h = 2$ | | $d/h = 4$ | |
|---|---|---|---|---|
| $(h-b)/h$ | $k^*(b)$ | $\bar{k}^*(b)$ | $k^*(b)$ | $\bar{k}^*(b)$ |
| 0.01 | 6.345 | 6.434 | 13.00 | 13.09 |
| 0.10 | 5.883 | 5.910 | 12.17 | 12.19 |
| 0.20 | 5.867 | 5.882 | 12.20 | 12.22 |
| 0.30 | 6.243 | 6.255 | 12.99 | 13.01 |
| 0.40 | 7.041 | 7.042 | 14.61 | 14.60 |
| 0.50 | 8.448 | 8.467 | 17.44 | 17.46 |
| 0.60 | 10.92 | 10.96 | 22.42 | 22.45 |
| 0.70 | 15.77 | | 32.12 | |
| 0.80 | 27.40 | | 55.51 | |

Table 2. Effect of the stamp width $2a$ on the stress intensity factor in a strip with an edge crack. $d/h = 2$, $(h-b)/h = 0.1$

| $a/h$ | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 |
|---|---|---|---|---|---|---|---|---|
| $k^*(b)$ | 5.883 | 5.876 | 5.856 | 5.819 | 5.762 | 5.681 | 5.570 | separat. |

Table 3. Stress intensity factor vs. the crack length for an edge crack. $a/h = 0.2$, $d/h = 4.0$

| $(h-b)/h$ | 0.01 | 0.10 | 0.20 |
|---|---|---|---|
| $k^*(b)$ | 12.84 | 12.03 | separation |

Table 4. Stress intensity factors in a strip with a central crack. $a/h = 0.01$, $d/h = 4$, $c+b = h$, $k_0 = k/[(2P/h)\sqrt{((c-b)/2)}]$

| $(c-b)/h$ | $k_0(b)$ | $k_0(c)$ |
|---|---|---|
| 0.1 | −0.28763 | 0.78919 |
| 0.2 | −0.82050 | 1.3365 |
| 0.3 | −1.3540 | 1.8954 |
| 0.4 | −1.8975 | 2.4780 |
| 0.5 | −2.4714 | 3.1095 |
| 0.6 | −3.1177 | 3.8405 |
| 0.7 | −3.9294 | 4.7822 |
| 0.8 | −5.1552 | 6.2281 |
| 0.9 | −7.8157 | 9.3547 |
| 0.95 | −11.6675 | 13.7382 |

to the case of edge crack. Figure 3 gives the pressure distribution for a fixed crack length and for selected values of the stamp width $2a$. It is seen that as $a/h$ increases the pressure in the mid portion of the contact area decreases, and at approximately $a/h = 0.315$, $p(0)$ becomes zero. Upon further increasing $a/h$ the analysis gives negative pressure around $x = 0$. Since this is not possible separation would have to take place along the contact area. Similar results may be observed in Fig. 4 where the pressure distribution for a fixed stamp width and variable crack length is given. Here the separation begins approximately at $(h-b)/h = 0.15$. These results indicate that for a given crack length and stamp widths greater than a certain critical value or for a given stamp width and crack lengths greater than a certain value the solution as outlined in this paper would not be applicable. This is a typical "receding contact" problem in which the contact area is not known. However, our unpublished results in connection with the problem described in[13]† shows that in this case the contact area would be confined near the ends of the stamp and a very good approximation to the solution may be obtained from eqn (11) by replacing the contact pressure $p$ by two concentrated loads $P\delta(x-a)$ and $P\delta(x+a)$.

The values of the stress intensity factor corresponding to Figs. 3 and 4 are given in Tables 2 and 3.

The results for an internal crack $(0 < b < y < c < h)$ are shown in Tables 4 and 5. Table 4 shows the stress intensity factor ratio $k_0$ for a symmetrically located crack, i.e. for $(c+b)/2 = h/2$. Here the crack tip $y = b$ is in the compression region, hence $k(b) < 0$. Of course, considered separately these results are meaningless. However, the results can be used if the strip is also under a sufficiently large axial load so that in the combined bending-membrane

Table 5. Stress intensity factors for an eccentrically located crack. $a/h = 0.01$, $c+b = 3h/2$, $k_0 = k/[(2P/h)\sqrt{((c-b)/2)}]$

| | $d/h = 2$ | | $d/h = 4$ | |
|---|---|---|---|---|
| $(c-b)/h$ | $k_0(b)$ | $k_0(c)$ | $k_0(b)$ | $k_0(c)$ |
| 0.05 | 2.6122 | 2.8784 | 5.4749 | 6.0423 |
| 0.10 | 2.5167 | 3.0562 | 5.2663 | 6.4155 |
| 0.20 | 2.3986 | 3.5429 | 4.9947 | 7.4265 |
| 0.30 | 2.4018 | 4.3492 | 4.9681 | 9.0895 |
| 0.40 | 2.6470 | 6.1001 | 5.4362 | 12.6981 |

†The plane problem for an elastic strip supported by two elastic quarter planes and subjected to transverse loads. In this problem, upon separation, the contact area recedes towards the ends of the stamp and the solution becomes indistinguishable from the concentrated load solution.

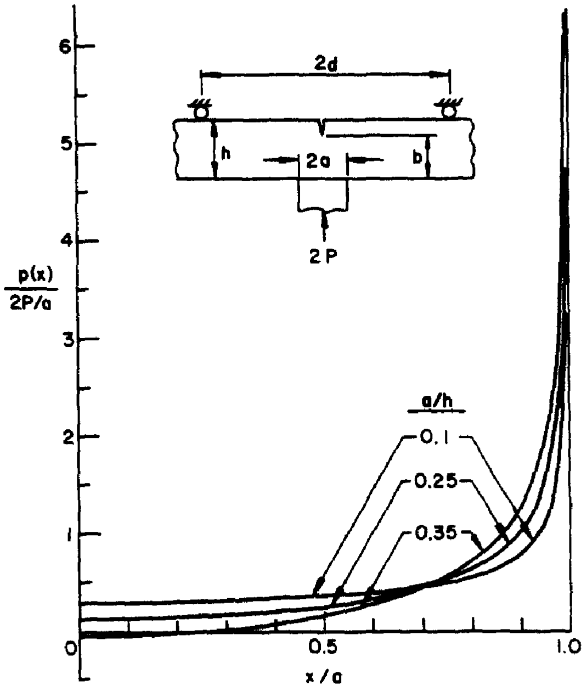


Fig. 3. Normalized pressure distribution under a flat rigid stamp with sharp edges in a strip with an edge crack of length $h-b$. $(h-b)/h = 0.1$, support spacing $= 2d = 4h$, stamp width $= 2a$, total compressive force $= 2P$.

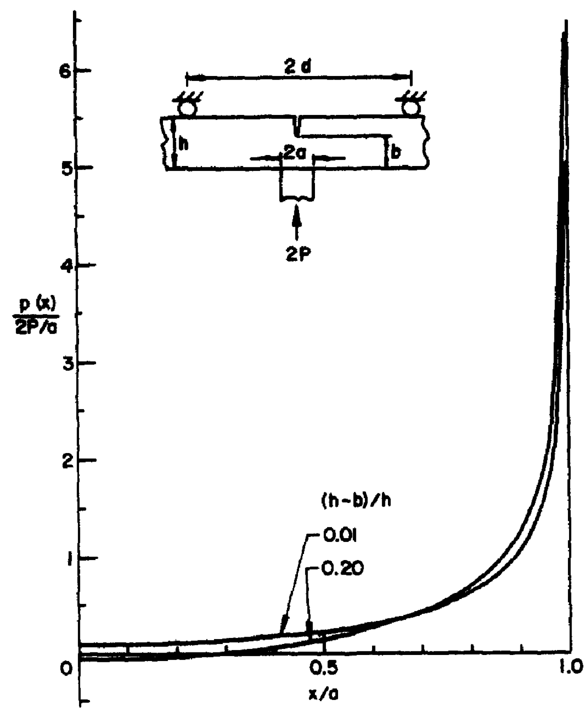


Fig. 4. Normalized pressure distribution under a flat rigid stamp with sharp edges in a strip with an edge crack. $a/h = 0.2$, $d/h = 4$, total force $2P$.

solution $k(b)$ becomes positive or zero. Then the results given in Table 4 would be quite useful. Note that in this problem $|k(b)| \neq |k(c)|$, whereas under pure bending one has $k(b) = -k(c)$. Table 5 shows some sample results for an eccentrically located crack.

## 4. ELASTIC STAMP

Let the strip again be supported at $x = \mp d$, $y = h$ and be loaded through a curved elastic stamp with a local radius of curvature $R$. If $R$ is large in comparison with the contact length $2a$, then the input function $V$ appearing in (10) may be expressed by

$$V(x) = -\frac{x}{R}. \tag{36}$$

The system of singular integral equation are again valid with

$$q(x) = 0, \quad Q(x) = 0, \quad P(x) = P\delta(x-d) + P\delta(x+d);$$
$$L = (b, c), \quad M_0 = (-a, a), \quad \sigma_{xx}(0, y) = 0, \quad b < y < c;$$
$$\int_b^c f(t)\,\mathrm{d}t = 0, \quad \int_{-a}^{a} p(t)\,\mathrm{d}t = 2P. \tag{37}$$

Using the transformation (23) and defining

$$\frac{p(as_1)}{\beta_3(a/R)} = g_1(s_1), \qquad \frac{f(t)}{a/R} = g_2(s_2), \tag{38}$$

the integral equations can be expressed in the following form

$$\int_{-1}^{1} \sum_{j=1}^{2} h_{ij}(r, s) g_j(s)\,\mathrm{d}s = v_i(r), \quad -1 < r < 1, \quad v_1(r) = -\pi r, \quad v_2(r) = 0, \quad i = 1, 2, \tag{39}$$

subject to the conditions

$$\int_{-1}^{1} g_1(s)\,\mathrm{d}s = \frac{R}{\beta_3 a^2} 2P = \lambda, \qquad \int_{-1}^{1} g_2(s)\,\mathrm{d}s = 0. \tag{40a, b}$$

In this problem the contact width $2a$ is unknown. The integral equations (39) are solved by assuming that $a$ is known. After determining $g_1(s)$ for a given $a$, the corresponding load $P$ is then determined from (40a). The numerical solution is obtained by letting

$$g_1(s) = G_1(s)(1-s^2)^{1/2}, \quad g_2(s) = G_2(s)(1-s^2)^{-1/2} \tag{41a, b}$$

and using the Gauss–Chebyshev integration formulas [10]. The stress intensity factors are then obtained from

$$k(b) = \frac{a}{R} G_2(-1)\sqrt{[(c-b)/2]}, \quad k(c) = -\frac{a}{R} G_2(1)\sqrt{[(c-b)/2]}. \tag{42a, b}$$

The calculated results obtained for the loading by a curved stamp are summarized in Table 6. The results are given for three values of $\beta_2$, namely $\beta_2$, namely $\beta_2 = 1$, $\beta_2 = 0.5$ and $\beta_2 = 5 \times 10^{-5}$ which correspond respectively to a rigid stamp, a stamp having the same elastic constants as that of the strip, and a stamp with a shear modulus which is much lower than that of the strip. The difference in these three sets of results lies mainly in the value of $\lambda$ as defined by (40a). Note that in terms of $\lambda$ and $k^*$ the actual stress intensity factor is given as

$$k(b) = \frac{a^2}{hR} \beta_3 k^*(b) \lambda \sqrt{(h-b)}. \tag{43}$$

Table 6. Stress intensity factors in an infinite strip with an edge crack loaded through an elastic curved stamp. $d/h = 4$

| $(h-b)/h$ | $a/h$ | $k^*(b)$ | $10^4\lambda a^2/h^2$ |
|---|---|---|---|
| | $\beta_2 = 1.0$ | | |
| 0.1 | 0.01 | 12.17 | 1.567 |
| 0.1 | 0.03 | 12.17 | 13.87 |
| 0.1 | 0.05 | 12.17 | 37.25 |
| 0.1 | 0.10 | 12.16 | 129.2 |
| 0.1 | 0.15 | 12.14 | 238.3 |
| 0.5 | 0.01 | 17.44 | 1.549 |
| 0.5 | 0.03 | 17.44 | 12.54 |
| 0.5 | 0.05 | 17.44 | 29.02 |
| 0.7 | 0.01 | 32.11 | 1.469 |
| | $\beta_2 = 0.5$ | | |
| 0.1 | 0.01 | 12.17 | 1.569 |
| 0.1 | 0.03 | 12.17 | 14.00 |
| 0.1 | 0.10 | 12.16 | 141.8 |
| 0.1 | 0.20 | 12.12 | 440.3 |
| 0.5 | 0.03 | 17.44 | 13.29 |
| 0.5 | 0.05 | 17.44 | 33.38 |
| 0.5 | 0.10 | 17.42 | 92.81 |
| 0.5 | 0.20 | 17.34 | 175.9 |
| 0.7 | 0.03 | 32.11 | 10.80 |
| 0.7 | 0.05 | 32.09 | 21.27 |
| 0.7 | 0.10 | 32.03 | 37.07 |
| | $\beta_2 = 5.0 \times 10^{-5}$ | | |
| 0.1 | 0.10 | 12.16 | 157.1 |
| 0.1 | 0.50 | 11.88 | 3,926 |
| 0.1 | 1.0 | 11.27 | 15,690 |
| 0.5 | 0.10 | 17.42 | 157.1 |
| 0.5 | 0.50 | 16.94 | 3,923 |
| 0.5 | 1.0 | 16.05 | 15,660 |

## 5. THE EFFECT OF FRICTION AT THE SUPPORTS

If the beam or the plate is loaded as described in Fig. 1 and if there is friction at the supports, then the axial load $Q$ would not be zero and may tend to increase the stress intensity factor. Or, in general, the strip may be subjected to axial surface loads $Q$ not necessarily at the supports. Assuming that $P$ and $Q$ are known, the integral equations (10) and (11) can be solved and the effect of $Q$ can be evaluated. As an example, in this paper it is assumed that $Q$ is a known concentrated force at the supports. Hence (10) and (11) are solved by simply letting $Q = \eta P$, where $\eta$ is a known coefficient.

Table 7 shows the results of a numerical example in which it is assumed that $Q = P$. This is basically a three point loading problem with friction at the supports. The normalized stress intensity factor is again defined by

$$k_Q^*(b) = k(b)/[(2P/h)\sqrt{(h-b)}]. \tag{44}$$

The table also shows the difference $\Delta k$ between the stress intensity factors calculated with and without taking the effect of $Q$ into account (for $k^*(b)$ see Table 1).

Table 7. Effect of friction at the supports on the stress intensity factor in a strip with an edge crack loaded through a flat rigid stamp. $a/h = 0.01$, $d/h = 2$, $Q = P$

| $(h-b)/h$ | 0.01 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| $k_Q^*(b)$ | 8.570 | 8.049 | 8.135 | 8.761 | 9.998 |
| $k_Q^*(b) - k^*(b)$ | 2.225 | 2.166 | 2.268 | 2.518 | 2.947 |

Table 8. Normalized pressure distribution under a flat rigid stamp in a strip with an edge crack of length $h-b$ (Fig. 3). $(h-b)/h = 0.1$, $d = 2h$, $2a$: stamp width, $2P$: total compressive force

| | $p(x)/(2P/a)$ | | |
|---|---|---|---|
| $x/a$ | $a/h = 0.1$ | $a/h = 0.01$ | $(a/h)\to 0$ |
| 0.054 | 0.285 | 0.318 | 0.3187 |
| 0.162 | 0.290 | 0.322 | 0.3226 |
| 0.267 | 0.301 | 0.330 | 0.3303 |
| 0.370 | 0.316 | 0.342 | 0.3426 |
| 0.468 | 0.339 | 0.360 | 0.3602 |
| 0.561 | 0.370 | 0.384 | 0.3845 |
| 0.647 | 0.411 | 0.417 | 0.4175 |
| 0.726 | 0.465 | 0.463 | 0.4629 |
| 0.796 | 0.541 | 0.526 | 0.5259 |
| 0.857 | 0.648 | 0.618 | 0.6177 |
| 0.907 | 0.809 | 0.758 | 0.7558 |
| 0.948 | 1.080 | 0.998 | 1.0001 |
| 0.977 | 1.621 | 1.482 | 1.4927 |
| 0.994 | 3.245 | 2.947 | 2.9101 |

One may estimate the effect of the friction for values of $\eta$ other than unity by simply assuming that $\Delta k$ is proportional to $\eta$. The table indicates that as one would expect the effect of friction or any axial constraint may be significant in transversely loaded beams and plates.

In the crack-contact problems described in this paper for a fixed value of $b/h$ as $a/h$ tends to zero the (relative) pressure distribution under the stamp would approach the contact stress between a stamp and a semi-infinite elastic medium. Table 8 shows the result of an example indicating this trend. The example is that considered in Fig. 3. The pressure distribution corresponding to $(a/h)\to 0$ shown in Table 8 is the contact stress for a frictionless rigid flat stamp pressed on an elastic half plane and is given by

$$\frac{p(x)}{2P/a} = \frac{1}{\pi\sqrt{[1-(x/a)^2]}}, \qquad \int_{-a}^{a} p(x)\,dx = 2P. \tag{45}$$

In a graph giving the pressure distribution such as Fig. 3 the differences shown in Table 8 would be too small to be distinguishable.

A brief remark regarding the numerical analysis may also be useful. For the internal crack, depending on the distances of the crack tips to the boundaries, $n = 15$ in the algebraic equations (30) was sufficient to obtain three or four digit accuracy in the stress intensity factors. However, for the edge crack, particularly for deep cracks, greater number of equations was needed to obtain four digit accuracy. To have some idea about the number of equations used in the solution consider, for example, the results given in Table 1. In order to obtain the four digit accuracy shown in the table the numbers $n$ shown in Table 9 had to be used. For $(h-b)/h \geq 0.6$ (i.e. for deep cracks) $n = 60$ gave only a three digit accuracy. The stress intensity factors shown in the table for crack depth ratios 0.6, 0.7 and 0.8 were obtained by extrapolating the results found from $n = 40$, $n = 50$, and $n = 60$. For this the stress intensity factor was expressed as

$$k^*(n) = A + B/n^{\alpha} \tag{46}$$

Table 9. Number of equations $n$ used to compute the values given in Table 1, Extp.: extrapolation from $n = 40, 50, 60$

| $\frac{h-b}{h}$ | 0.01 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| $n$ $(d = 2h)$ | 30 | 30 | 30 | 30 | 40 | 45 | Extp. | Extp. | Extp. |
| $n$ $(d = 4h)$ | 30 | 35 | 35 | 35 | 40 | 50 | Extp. | Extp. | Extp. |

where the constants $A$, $B$ and $\alpha$ were obtained by using the values found for $n = 40$, 50 and 60. The constant $A$ corresponding to $n \to \infty$ is the value shown in the table. Even though, in this example the same number $n$ was used in both equations given by (30), from the viewpoint of numerical analysis the stamp is equivalent to an internal crack. This was shown to be the case numerically and in the subsequent examples the number of equations used in (30) for $i = 1$ and $i = 2$ were taken to be different, $n_1$ being less than $n_2$.

## REFERENCES


1. W. T. Koiter, Rectangular tensile sheet with symmetric edge cracks. *J. Appl. Mech., Trans. ASME* **32**, 237 (1965).
2. J. P. Benthem and W. T. Koiter, Asymptotic approximations to crack problems. In *Methods of Analysis and Solutions of Crack Problems* (Edited by G. C. Sih), p. 131. Noordhoff, Leyden (1973).
3. I. N. Sneddon and R. P. Srivastav, The stress field in the vicinity of a Griffith crack in a strip of finite width. *Int. J. Engng Sci.* **9**, 479 (1971).
4. G. D. Gupta and F. Erdogan, The problem of edge cracks in an infinite strip. *J. Appl. Mech., Trans. ASME* **41**, 1001 (1974).
5. B. Gross and J. E. Srawley, Stress Intensity Factors for Three Point Bend Specimens by Boundary Collocation. *NASA Tech. Note D*-3092 (Dec. 1965).
6. M. Isida, Method of Laurent series expansion for internal crack problems. In *Methods of Analysis and Solutions of Crack Problems* (Edited by G. C. Sih), p. 56. Noordhoff, Leyden (1973).
7. O. L. Bowie, Solutions of plane crack problems by mapping techniques. In *Methods of Analysis and Solutions of Crack Problems* (Edited by G. C. Sih), p. 1. Noordhoff, Leyden (1973).
8. M. Ratwani and F. Erdogan, On the plane contact problem for a frictionless elastic layer. *Int. J. Solids Structures* **9**, 921 (1973).
9. L. A. Galin, Contact Problems in the Theory of Elasticity (English Trans.), North Carolina State College, Raleigh, N.C. (1961).
10. F. Erdogan, Mixed Boundary—Value Problems in Mechanics. In *Mechanics Today* (Edited by S. Nemat-Nasser), Vol. 4, p. 1. Pergamon Press, Oxford (1978).
11. P. S. Theocaris and N. I. Ioakimidis, Numerical integration methods for the solution of singular integral equations. *Q. Appl. Math.* **35**, 173–183 (1977).
12. S. Krenk, Quadrature formulae of closed type for solution of singular integal equations. *J. Inst. Maths. Applic.* **22**, 99–107 (1978).
13. F. Erdogan and M. Ratwani, The contact problem for an elastic layer supported by two elastic quarter planes. *J. Appl. Mech. Trans. ASME* **41**, 673 (1974).


## APPENDIX A

The kernels $k_1, \ldots, k_{10}$

$$k_1(x,t) = \int_0^\infty B(\alpha)[4\alpha^2h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}] \sin \alpha(t-x)\, d\alpha,$$

$$k_2(x,t) = \int_0^\infty B(\alpha) 4\alpha^2h^2 \cos \alpha(t-x)\, d\alpha,$$

$$k_3(x,t) = -2\int_0^\infty B(\alpha)[(1+\alpha h)e^{\alpha h} - (1-\alpha h)e^{-\alpha h}] \sin \alpha(t-x)\, d\alpha,$$

$$k_4(x,t) = 2\int_0^\infty B(\alpha)\alpha h(e^{\alpha h} - e^{-\alpha h}) \cos \alpha(t-x)\, d\alpha,$$

$$\begin{aligned} k_5(x,t) = \int_0^\infty B(\alpha)\{ & e^{-\alpha t}[(1-\alpha t)4\alpha^2h^2 - \alpha t(4\alpha^2h^2 + 4\alpha h + 2 - 2e^{-2\alpha h})] \\ & + 2e^{-\alpha(h-t)}[(1 - \alpha h + \alpha t)\alpha h(e^{\alpha h} - e^{-\alpha h}) \\ & - \alpha(h-t)(e^{\alpha h}(1+\alpha h) - (1-\alpha h)e^{-\alpha h})]\} \sin \alpha x\, d\alpha, \end{aligned}$$

$$B(\alpha) = (e^{2\alpha h} - 4\alpha^2h^2 - 2 + e^{-2\alpha h})^{-1},$$

$$k_6(y,t) = \sum_1^{16} h_i(y,t),$$

$$h_1 = \int_0^\infty B(\alpha)(4\alpha^2h^2 + 4\alpha h + 2 - 2e^{-2\alpha h})\alpha t\, e^{-\alpha t} \sinh \alpha y\, d\alpha,$$

$$h_2 = \int_0^\infty B(\alpha)(\alpha t - 1)4\alpha^2h^2\, e^{-\alpha t} \sinh \alpha y\, d\alpha,$$

$$h_3 = -\int_0^\infty B(\alpha)4\alpha^4h^2 ty\, e^{-\alpha t} \sinh \alpha y\, d\alpha,$$

$$h_4 = \int_0^\infty B(\alpha)\alpha y(\alpha t - 1)(4\alpha h - 4\alpha^2h^2 - 2 + 2e^{-2\alpha h})\, e^{-\alpha t} \sinh \alpha y\, d\alpha,$$

$$\begin{aligned} h_5 = \int_0^\infty \alpha(h-t)\, e^{-\alpha(h-t)}\{ & B(\alpha)[2(1+\alpha h)\, e^{\alpha h} + 2(\alpha h - 1)\, e^{-\alpha h}] \\ & - 2(1+\alpha h)\, e^{-\alpha h}\} \sinh \alpha y\, d\alpha, \end{aligned}$$

$$h_6 = \int_0^\infty (1 - \alpha h + \alpha t)\, e^{-\alpha(h-t)}[B(\alpha)2\alpha h(e^{-\alpha h} - e^{\alpha h}) + 2\alpha h\, e^{-\alpha h}] \sinh \alpha y \, d\alpha$$

$$h_7 = \int_0^\infty y\alpha^2(t - h)\, e^{-\alpha(h-t)}[B(\alpha)2\alpha h(e^{\alpha h} - e^{-\alpha h}) - 2\alpha h\, e^{-\alpha h}] \sinh \alpha y \, d\alpha$$

$$h_8 = \int_0^\infty y\alpha(1 - \alpha h + \alpha t)\, e^{-\alpha(h-t)}\{2(1 - \alpha h)\, e^{-\alpha h} - B(\alpha)[2(1 - \alpha h)\, e^{\alpha h} - 2(1 + \alpha h)\, e^{-\alpha h}]\} \sinh \alpha y \, d\alpha,$$

$$h_9 = -\int_0^\infty B(\alpha) 8\alpha^3 h^2 t\, e^{-\alpha t} \cosh \alpha y \, d\alpha,$$

$$h_{10} = \int_0^\infty B(\alpha)(1 - \alpha t)\, e^{-\alpha t}(8\alpha^2 h^2 - 8\alpha h + 4 - 4e^{-2\alpha h}) \cosh \alpha y \, d\alpha,$$

$$h_{11} = \int_0^\infty B(\alpha)\alpha^2 y t\, e^{-\alpha t}(4\alpha^2 h^2 + 4\alpha h + 2 - 2e^{-2\alpha h}) \cosh \alpha y \, d\alpha,$$

$$h_{12} = \int_0^\infty B(\alpha) 4\alpha^3 h^2 (\alpha t - 1) y\, e^{-\alpha t} \cosh \alpha y \, d\alpha,$$

$$h_{13} = \int_0^\infty 4\alpha^2 h[e^{-\alpha h} - B(\alpha)(e^{\alpha h} - e^{-\alpha h})](h - t)\, e^{-\alpha(h-t)} \cosh \alpha y \, d\alpha,$$

$$h_{14} = \int_0^\infty 4(1 - \alpha h + \alpha t)\, e^{-\alpha(h-t)}\{(1 - \alpha h)\, e^{-\alpha h} + B(\alpha)[(\alpha h - 1)\, e^{\alpha h} + (\alpha h + 1)\, e^{-\alpha h}]\} \cosh \alpha y \, d\alpha,$$

$$h_{15} = \int_0^\infty 2\alpha^2 y(h - t)\, e^{-\alpha(h-t)}\{B(\alpha)[(1 + \alpha h)\, e^{\alpha h} + (\alpha h - 1)\, e^{-\alpha h}] - (1 + \alpha h)\, e^{-\alpha h}\} \cosh \alpha y \, d\alpha,$$

$$h_{16} = \int_0^\infty 2\alpha^2 h y(1 - \alpha h + \alpha t)\, e^{-\alpha(h-t)}[B(\alpha)(e^{-\alpha h} - e^{\alpha h}) + e^{-\alpha h}] \cosh \alpha y \, d\alpha.$$

$$k_7(y, t) = \sum_1^4 a_i(y, t)$$

$$a_1 = \int_0^\infty B(\alpha)(2\alpha h + 1 - 2\alpha^2 h^2 - e^{-2\alpha h})\, e^{\alpha y} \cos \alpha t \, d\alpha,$$

$$a_2 = \int_0^\infty B(\alpha)(e^{-2\alpha h} - 1 - 2\alpha h - 6\alpha^2 h^2)\, e^{-\alpha y} \cos \alpha t \, d\alpha,$$

$$a_3 = \int_0^\infty B(\alpha)\alpha y(2\alpha h + 1 - e^{-2\alpha h})\, e^{\alpha y} \cos \alpha t \, d\alpha,$$

$$a_4 = \int_0^\infty B(\alpha)\alpha y(4\alpha^2 h^2 + 2\alpha h + 1 - e^{-2\alpha h})\, e^{-\alpha y} \cos \alpha t \, d\alpha,$$

$$k_8(y, t) = \sum_1^4 b_i(y, t),$$

$$b_1 = \int_0^\infty 2B(\alpha)(2\alpha h - \alpha^2 h^2 - 1 + e^{-2\alpha h})\, e^{\alpha y} \sin \alpha t \, d\alpha,$$

$$b_2 = \int_0^\infty 2B(\alpha)(2\alpha h - 3\alpha^2 h^2 - 1 + e^{-2\alpha h})\, e^{-\alpha y} \sin \alpha t \, d\alpha,$$

$$b_3 = \int_0^\infty B(\alpha)\alpha y(2\alpha h - 1 + e^{-2\alpha h})\, e^{\alpha y} \sin \alpha t \, d\alpha,$$

$$b_4 = \int_0^\infty B(\alpha)\alpha y(4\alpha^2 h^2 - 2\alpha h + 1 - e^{-2\alpha h})\, e^{-\alpha y} \sin \alpha t \, d\alpha.$$

$$k_9(y, t) = \sum_1^4 c_i(y, t),$$

$$c_1 = \int_0^\infty \{B(\alpha)[(\alpha h - 1)\, e^{\alpha h} + (1 - 3\alpha h)\, e^{-\alpha h}] - (\alpha h - 1)\, e^{-\alpha h}\}\, e^{\alpha y} \cos \alpha t \, d\alpha,$$

$$c_2 = \int_0^\infty B(\alpha)[(1 + 3\alpha h)\, e^{\alpha h} - (1 + \alpha h)\, e^{-\alpha h}]\, e^{-\alpha y} \cos \alpha t \, d\alpha,$$

$$c_3 = \int_0^\infty \{B(\alpha)[(1 - 2\alpha h)\, e^{-\alpha h} - e^{\alpha h}] + e^{-\alpha h}\}\, \alpha y\, e^{\alpha y} \cos \alpha t \, d\alpha,$$

$$c_4 = \int_0^\infty B(\alpha)[e^{-\alpha h} - (1 + 2\alpha h)\, e^{\alpha h}]\alpha y\, e^{-\alpha y} \cos \alpha t \, d\alpha.$$

$$k_{10}(y, t) = \sum_1^4 d_i(y, t),$$

$$d_1 = \int_0^\infty \{B(\alpha)[(2 - \alpha h)\, e^{\alpha h} - (2 + 3\alpha h)\, e^{-\alpha h}] - (2 - \alpha h)\, e^{-\alpha h}\}\, e^{\alpha y} \sin \alpha t \, d\alpha,$$

$$d_2 = \int_0^\infty B(\alpha)[(2-3\alpha h)\,e^{\alpha h} - (2+\alpha h)\,e^{-\alpha h}]\,e^{-\alpha y}\sin\alpha t\,d\alpha,$$

$$d_3 = \int_0^\infty \{B(\alpha)[e^{\alpha h} - (1+2\alpha h)\,e^{-\alpha h}] - e^{-\alpha h}\}\,\alpha y\,e^{\alpha y}\sin\alpha t\,d\alpha,$$

$$d_4 = \int_0^\infty B(\alpha)[(2\alpha h - 1)\,e^{\alpha h} + e^{-\alpha h}]\,\alpha y\,e^{-\alpha y}\sin\alpha t\,d\alpha.$$